LE MATEMATICHE
Vol. LIII (1998) – Fasc. I, pp. 51–60

ON SZASZ-MIRAKYAN OPERATORS OF FUNCTIONS OF TWO VARIABLES

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We consider Szasz-Mirakyan operators $S_{m,n}^{\{i\}}$ in polynomial and exponential weighted spaces of functions of two variables. We give Voronowskaya type theorem and theorem on convergence of sequence $\left\{\frac{\partial}{\partial x} S_{n,n}^{\{i\}}(f)\right\}$.

1. Preliminaries.

1.1. Similarly as in [1] and [2], for fixed $p \in N_0 := \{0, 1, 2, \dots\}$ and $q \in R_+ := (0, +\infty)$ and for all $x \in R_0 := R_+ \cup \{0\}$, we define

(1) $$w_0(x) := 1, \qquad w_p(x) := (1 + x^p)^{-1} \qquad \text{if} \qquad p \geq 1,$$

(2) $$v_q(x) := e^{-qx}.$$

Next, for fixed $p_1, p_2 \in N_0$, we define the weighted function

(3) $$w_{p_1,p_2}(x, y) := w_{p_1}(x) w_{p_2}(y), \qquad (x, y) \in R_0^2 := R_0 \times R_0,$$

and the polynomial weighted space $C_{1;p_1,p_2}$ of real-valued functions f continuous on R_0^2 for which $f w_{p_1,p_2}$ is uniformly continuous and bounded on R_0^2. The norm in $C_{1;p_1,p_2}$ is defined by

(4) $$\|f\|_{1;p_1,p_2} := \sup_{(x,y)\in R_0^2} w_{p_1,p_2}(x, y)\left|f(x, y)\right|.$$

Entrato in Redazione il 17 dicembre 1997.
AMS Subject Classification: 41A36.
Key words: Szasz-Mirakyan operator, Weighted space, Function of two variables.

Analogously, for fixed $q_1, q_2 \in R_+$, we define

$$v_{q_1,q_2}(x, y) := v_{q_1}(x)v_{q_2}(y), \qquad (x, y) \in R_0^2, \tag{5}$$

and the exponential weighted space $C_{2;q_1q_2}$ of real-valued functions f continuous on R_0^2 for which $f v_{q_1,q_2}$ is uniformly continuous and bounded on R_0^2. The norm in $C_{2;q_1,q_2}$ is given by

$$\|f\|_{2;q_1,q_2} := \sup_{(x,y)\in R_0^2} v_{q_1,q_2}(x, y)\big|f(x, y)\big|. \tag{6}$$

Moreover, for fixed $m \in N := \{1, 2, \ldots\}$ and $p_1, p_2 \in N_0$, let $C^m_{1;p_1,p_2}$ be the class of all functions $f \in C_{1;p_1,p_2}$ which partial derivatives of the order $\leq m$ belong to $C_{1;p_1,p_2}$ also. Analogously we define the class $C^m_{2;q_1,q_2}$, $m \in N$ and $q_1, q_2 \in R_+$.

1.2. In [3] were examined the Szasz-Mirakyan operators for functions f continuous on R_0^2

$$S^{\{1\}}_{m,n}(f; x, y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)f\Big(\frac{j}{m}, \frac{k}{n}\Big), \tag{7}$$

$$S^{\{2\}}_{m,n}(f; x, y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)mn \int_{\frac{j}{m}}^{\frac{j+1}{m}}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t, z)\, dt dz, \tag{8}$$

$(x, y) \in R_0^2$, $m, n \in N$, where

$$a_{n,k}(x) := e^{-nx}\frac{(nx)^k}{k!}, \qquad x \in R_0,\ k \in N_0,\ n \in N. \tag{9}$$

These operators are analogues of the Szasz-Mirakyan operators, considered in [1] – [3] for functions f of one variable

$$S^{\{1\}}_{n}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)f\Big(\frac{k}{n}\Big), \tag{10}$$

$$S^{\{2\}}_{n}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \quad x \in R_0,\ n \in N. \tag{11}$$

From the results given in [3] we can deduce that if $f \in C_{1;p_1,p_2}$ or $f \in C_{2;q_1,q_2}$, with some $p_1, p_2 \in n_0$ and $q_1, q_2 \in R_+$, then

$$\lim_{m,n\to\infty} S^{\{i\}}_{m,n}(f; x, y) = f(x, y), \qquad i = 1, 2, \tag{12}$$

for every $(x, y) \in R_0^2$.

In the present paper we shall prove some analogues of (12) for derivatives of the operators (7) and (8). In Section 2 we shall give some auxiliary results and in Section 3 we shall prove the main theorems.

By $M_k(a, b)$, $k = 1, 2, \ldots$, we shall denote suitable positive constants depending only on indicated parameters a, b. The partial derivative of function f we shall denote as usual by f'_x or $\frac{\partial f}{\partial x}$.

2. Auxiliary results.

2.1. First we shall give some properties of the operators $S^{\{i\}}_n$ and $S^{\{i\}}_{m,n}$ proved in [1] – [3].

From (7) – (11) it follows that

$$S^{\{i\}}_n(1; x) = 1, \qquad x \in R_0,\ n \in N,\ i = 1, 2, \tag{13}$$

$$S^{\{i\}}_{m,n}(1; x, y) = 1, \qquad (x, y) \in R_0^2,\ m, n \in N,\ i = 1, 2. \tag{14}$$

Moreover, if $f \in C_{1;p_1,p_2}$ or $f \in C_{2;q_1,q_2}$ $(p_1, p_2 \in N_0,\ q_1, q_2 \in R_+)$ and if $f(x, y) = f_1(x) f_2(y)$ for all $(x, y) \in R_0^2$, then

$$S^{\{i\}}_{m,n}(f(t, z); x, y) = S^{\{i\}}_m(f_1(t); x) S^{\{i\}}_n(f_2(z); y) \tag{15}$$

for all $(x, y) \in R_0^2$, $m, n \in N$ and $i = 1, 2$.

For every fixed $x \in R_0$ and for all $n \in N$ we have ([1])

$$S^{\{i\}}_n(t - x; x) = \begin{cases} 0 & \text{if} \quad i = 1, \\ \dfrac{1}{2n} & \text{if} \quad i = 2, \end{cases} \tag{16}$$

$$S^{\{i\}}_n((t - x)^2; x) = \begin{cases} \dfrac{x}{n} & \text{if} \quad i = 1, \\ \dfrac{x}{n} + \dfrac{1}{3n^2} & \text{if} \quad i = 2. \end{cases} \tag{17}$$

Lemma 1 ([1]). *For every fixed $x_0 \in R_0$ there exists a positive constant $M_1(x_0)$ such that for all $n \in N$ and $i = 1, 2$*

$$S^{\{i\}}_n((t - x_0)^4; x_0) \le M_1(x_0) n^{-2}.$$

Lemma 2 ([1]). *For every fixed $p \in N_0$ there exists a positive constant $M_2(p)$ such that for all $x \in R_0$, $n \in N$ and $i = 1, 2$*

$$w_p(x) S_n^{\{i\}}(1/w_p(t); x) \leq M_2(p),$$

$$w_p(x) S_n^{\{i\}}\Big((t-x)^2/w_p(t); x\Big) \leq M_2(p) \begin{cases} \dfrac{x}{n} & \text{if} \quad i = 1, \\ \dfrac{x+1}{n} & \text{if} \quad i = 2. \end{cases}$$

Lemma 3 ([2]). *Let $r > q > 0$ are fixed numbers. Then there exist $M_3(q, r) =$ const > 0 and natural number $n_0 > q(ln(r/q))^{-1}$ such that for all $x \in R_0$, $n \geq n_0$ and $i = 1, 2$*

$$v_r(x) S_n^{\{i\}}(1/v_q(t); x) \leq M_3(q, r)$$

$$v_r(x) S_n^{\{i\}}((t-x)^2/v_q(t); x) \leq M_3(q, r) \begin{cases} \dfrac{x}{n} & \text{if} \quad i = 1, \\ \dfrac{x+1}{n} & \text{if} \quad i = 2. \end{cases}$$

Applying these lemmas, (1) – (6) and (15), we immediately derive from (7) – (9) the following two lemmas.

Lemma 4. *For fixed $p_1, p_2 \in N_0$ there exists $M_4(p_1, p_2) =$ const > 0 such that for every $f \in C_{1;p_1,p_2}$ and for all $m, n \in N$, $i = 1, 2$*

$$\|S_{m,n}^{\{i\}}(f; \cdot, \cdot)\|_{1;p_1,p_2} \leq M_4(p_1, p_2) \|f\|_{1;p_1,p_2}. \tag{18}$$

In particular

$$\|S_{m,n}^{\{i\}}(1/w_{p_1,p_2}(t, z); \cdot, \cdot)\|_{1;p_1,p_2} \leq M_4(p_1, p_2) \quad \text{for } m, n \in N, \ i = 1, 2. \tag{19}$$

From (7) – (9) *and* (18) *we deduce that $S_{m,n}^{\{i\}}$, $m, n \in N$, $i = 1, 2$, is a linear positive operator from the space $C_{1;p_1,p_2}$ into $C_{1;p_1,p_2}$.*

Lemma 5. *For fixed $r_1 > q_1 > 0$ and $r_2 > q_2 > 0$ there exist $M_5(q_1, q_2, r_1, r_2) =$ const > 0 and natural numbers $m_0 > q_1(ln(r_1/q_1))^{-1}$, $n_0 > q_2(l_n(r_2/q_2))^{-1}$ such that for all $m \geq m_0$, $n \geq n_0$ and $i = 1, 2$*

$$\|S_{m,n}^{\{i\}}(1/v_{q_1,q_2}(t, z); \cdot, \cdot)\|_{2;r_1,r_2} \leq M_5(q_1, q_2, r_1, r_2).$$

Moreover, for every $f \in C_{2;q_1,q_2}$ *and for all* $m \geq m_0$, $n \geq n_0$ *and* $i = 1, 2$ *we have*

$$\|S_{m,n}^{\{i\}}(f; \cdot, \cdot)\|_{2;r_1,r_2} \leq M_5(q_1, q_2, r_1, r_2)\|f\|_{2;q_1,q_2}. \tag{20}$$

The formulas (7) – (9) *and the inequality* (20) *prove that* $S_{m,n}^{\{i\}}$, $i = 1, 2$, *is a positive linear operator from the space* $C_{2;q_1,q_2}$ *into* $C_{2;r_1,r_2}$ *provided that* $r_1 > q_1 > 0$, $r_2 > q_2 > 0$ *and* $m \geq m_0$, $n \geq n_0$.

3. Main results.

3.1. First we shall prove the Voronovskaya type theorem.

Theorem 1. *Suppose that* $f \in C^2_{1;p_1,p_2}$ *or* $f \in C^2_{2;q_1,q_2}$ *with some* $p_1, p_2 \in N_0$, $q_1, q_2 \in R_+$. *Then, for every* $(x, y) \in R^2_+ := R_+ \times R_+$ *and* $i = 1, 2$, *we have*

$$\lim_{n\to\infty} n\{S_{n,n}^{\{i\}}(f; x, y) - f(x, y)\} = \frac{x}{2} f''_{xx}(x, y) + \frac{y}{2} f''_{yy}(x, y) + \tag{21}$$
$$+ \begin{cases} 0 & \text{if} \quad i = 1, \\ \frac{1}{2} f'_x(x, y) + \frac{1}{2} f'_y(x, y) & \text{if} \quad i = 2 \end{cases}.$$

Proof. Let $i = 1$, $f \in C^2_{1;p_1,p_2}$ and let $(x_0, y_0) \in R^2_+$ be fixed point. Then, by the Taylor formula, we can write

$$f(t, z) = f(x_0, y_0) + f'_x(x_0, y_0)(t - x_0) + f'_y(x_0, y_0)(z - y_0) +$$
$$+\frac{1}{2}\{f''_{xx}(x_0, y_0)(t-x_0)^2+2f''_{xy}(x_0, y_0)(t-x_0)(z-y_0)+f''_{yy}(x_0, y_0)(z-y_0)^2\}+$$
$$+ \varphi(t, z; x_0, y_0)\sqrt{(t - x_0)^4 + (z - y_0)^4}, \quad (t, z) \in R^2_0,$$

where $\varphi(t, z) \equiv \varphi(t, z; x_0, y_0)$ belongs to $C_{1;p_1,p_2}$ and $\lim_{(t,z)\to(x_0,y_0)} \varphi(t, z) = 0$. From this, applying (13) – (15), we get

$$S_{n,n}^{\{1\}}(f(t, z); x_0, y_0) = f(x_0, y_0) + \tag{22}$$
$$+ f'_x(x_0, y_0)S_n^{\{1\}}(t - x_0; x_0) + f'_y(x_0, y_0)S_n^{\{1\}}(z - y_0; y_0) +$$
$$+\frac{1}{2}\{f''_{xx}(x_0, y_0)S_n^{\{1\}}((t-x_0)^2; x_0)+2f''_{xy}(x_0, y_0)S_n^{\{1\}}(t-x_0; x_0)S_n^{\{1\}}(z-y_0; y_0)+$$
$$+ f''_{yy}(x_0, y_0)S_n^{\{1\}}((z-y_0)^2; y_0)\}+S_{n,n}^{\{1\}}(\varphi(t, z)\sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0),$$
$$n \in N.$$

But from (16) and (17) it follows that

$$\lim_{n\to\infty} nS_n^{\{1\}}(t-x_0;x_0)=0=\lim_{n\to\infty} nS_n^{\{1\}}(z-y_0;y_0), \tag{23}$$

$$\lim_{n\to\infty} nS_n^{\{1\}}((t-x_0)^2;x_0)=x_0 \quad , \quad \lim_{n\to\infty} nS_n^{\{1\}}((z-y_0)^2;y_0)=y_0. \tag{24}$$

By the Hölder inequality and by the linearity of $S_{n,n}^{\{1\}}$ and (13) – (15) we get

$$\left|S_{n,n}^{\{1\}}(\varphi(t,z)\sqrt{(t-x_0)^4+(z-y^0)^4};x_0,y_0)\right| \leq$$

$$\leq \{S_{n,n}^{\{1\}}(\varphi^2(t,z);x_0,y_0)\}^{1/2}\{S_n^{\{1\}}((t-x_0)^4;x_0)+S_n^{\{1\}}((z-y_0)^4;y_0)\}^{1/2},\ n\in N.$$

But by properties of φ and (12), we have

$$\lim_{n\to\infty} S_{n,n}^{\{1\}}(\varphi^2(t,z);x_0,y_0)=\varphi^2(x_0,y_0)=0.$$

From the foregoing facts and Lemma 1 we obtain

$$\lim_{n\to\infty} nS_{n,n}^{\{1\}}(\varphi(t,z)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0)=0. \tag{25}$$

Next, using (23) – (25), we derive from (22)

$$\lim_{n\to\infty} n\left\{S_{n,n}^{\{1\}}(f(t,z);x_0,y_0)-f(x_0,y_0)\right\}=\frac{x_0}{2}f''_{xx}(x_0,y_0)+\frac{y_0}{2}f''_{yy}(x_0,y_0).$$

Thus the proof of (21) for $i=1$ and $f\in C^2_{1;p_1,p_2}$ is completed. The proof of (21) in the other cases in analogous. $\square$

3.2. Now we shall give analogues of (12) for partial derivatives of $S_{n,n}^{\{i\}}(f;\cdot,\cdot)$.

Theorem 2. *Suppose that $f\in C^1_{1;p_1,p_2}$ or $f\in C^1_{2;q_1q_2}$ with some $p_1,p_2\in N_0$, $q_1,q_2\in R_+$. Then for every $(x,y)\in R^2_+$ and $i=1,2$*

$$\lim_{n\to\infty}\frac{\partial}{\partial x}S_{n,n}^{\{i\}}(f;x,y)=\frac{\partial f}{\partial x}(x,y), \tag{26}$$

$$\lim_{n\to\infty}\frac{\partial}{\partial y}S_{n,n}^{\{i\}}(f;x,y)=\frac{\partial f}{\partial y}(x,y). \tag{27}$$

Proof. We shall prove only (26), because the proof of (27) is identical. Let $i = 1$, $f \in C^1_{1;p_1,p_2}$ and let (x, y) be a fixed point in R^2_+. From (7) and (9) it follows that

$$\frac{\partial}{\partial x} S^{\{1\}}_{n,n}(f(t,z); x, y) = -nS^{\{1\}}_{n,n}(f(t,z); x, y) + \frac{n}{x} S^{\{1\}}_{n,n}(tf(t,z); x, y)$$

for every $n \in N$. Applying the Taylor formula for $f \in C^1_{1;p_1,p_2}$, we can write

$$\begin{aligned}(28)\qquad f(t,z) = f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(z-y) + \\ + \psi(t,z;x,y)\sqrt{(t-x)^2+(z-y)^2}, \quad (t,z) \in R^2_0,\end{aligned}$$

where $\psi(t,z) \equiv \psi(t,z;x,y)$ is function of the class $C_{1;p_1,p_2}$ and

$$\lim_{(t,z)\to(x,y)} \psi(t,z) = 0.$$

From the foregoing formulas and by (13) – (15) we get

$$\begin{aligned}\frac{\partial}{\partial x} S^{\{1\}}_{n,n}(f(t,z); x, y) = -n\big\{f(x,y) + f'_x(x,y)S^{\{1\}}_n(t-x;x) + \\ + f'_y(x,y)S^{\{1\}}_n(z-y;y) + S^{\{1\}}_{n,n}\big(\psi(t,z)\sqrt{(t-x)^2+(z-y)^2}; xy\big)\big\} + \\ + \frac{n}{x}\big\{f(x,y)S^{\{1\}}_n(t;x) + f'_x(x,y)S^{\{1\}}_n(t(t-x);x) + \\ + f'_y(x,y)S^{\{1\}}_n(t;x)S^{\{1\}}_n(z-y;y) + S^{\{1\}}_{n,n}(t\psi(t,z)\sqrt{(t-x)^2+(z-y)^2}; x, y\big\}, \\ n \in N,\end{aligned}$$

which by

$$S^{\{1\}}_n(t(t-x);x) = S^{\{1\}}_n((t-x)^2;x) + xS^{\{1\}}_n(t-x;x)$$

and by (16) and (17) implies

$$\begin{aligned}(29)\qquad \frac{\partial}{\partial x} S^{\{1\}}_{n,n}(f(t,z); x, y) = f'_x(x,y) + \\ + \frac{n}{x} S^{\{1\}}_{n,n}\Big((t-x)\psi(t,z)\sqrt{(t-x)^2+(z-y)^2}; x, y\Big)\end{aligned}$$

for all $n \in N$. Next, applying the Hölder inequality and (13) – (15), we have

$$|S_{n,n}^{\{1\}}((t-x)\psi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y)| \leq$$
$$\leq \left\{S_{n,n}^{\{1\}}(\psi^2(t,z);x,y)\right\}^{1/2}\left\{S_{n,n}^{\{1\}}\Big((t-x)^4+(t-x)^2(z-y)^2;x,y)\Big)\right\}^{1/2} =$$
$$=\left\{S_{n,n}^{\{1\}}(\psi^2(t,z);x,y)\right\}^{1/2}\left\{S_n^{\{1\}}((t-x)^4;x)+S_n^{\{1\}}((t-x^2);x)S_n^{\{1\}}((z-y)^2;y)\right\}^{1/2},$$
$$n \in N.$$

From the foregoing inequality and by (17), Lemma 1 and (12) we deduce that

$$\lim_{n\to\infty} nS_{n,n}^{\{1\}}((t-x)\psi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y)=0.$$

Hence, from (29) we obtain

$$\lim_{n\to\infty}\frac{\partial}{\partial x}S_{n,n}^{\{1\}}(f(t,z);x,y)=\frac{\partial f}{\partial x}(x,y)\quad \text{for } (x,y)\in R_+^2,$$

which completes the proof of (26) for $i=1$.

Let $f \in C^1_{1;p_1,p_2}$, $i=2$ and (x,y) be a fixed point in R_+^2. From (7) – (11) it follows that

$$\frac{\partial}{\partial x}S_{n,n}^{\{2\}}(f;x,y)=-nS_{n,n}^{\{2\}}(f;x,y)+$$
$$+\frac{n}{x}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{n,j}(x)a_{n,k}(y)jn\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t,z)\,dtdz.$$

Similarly as in the case $i=1$, by (28) and by (13) – (17), we get

$$\text{(30)}\quad \frac{\partial}{\partial x}S_{n,n}^{\{2\}}(f(t,z);x,y)=-n\Big\{f(x,y)+f'_x(x,y)S_n^{\{2\}}(t-x;x)+$$
$$+f'_y(x,y)S_n^{\{2\}}(z-y;y)+S_{n,n}^{\{2\}}\Big(\psi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\Big)\Big\}+$$
$$+\frac{n}{x}\Big\{f(x,y)S_n^{\{1\}}(t;x)+f'_x(x,y)\sum_{j=0}^{\infty}a_{n,j}(x)j\int_{\frac{j}{n}}^{\frac{j+1}{n}}(t-x)\,dt+$$

$$+ f'_y(x,y)S_n^{\{1\}}(t;x)S_n^{\{2\}}(z-y;y) +$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y)jn \int\limits_{\frac{j}{n}}^{\frac{j+1}{n}}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(t,z)\sqrt{(t-x)^2+(z-y)^2}\,dtdz\Bigg\} =$$

$$= f'_x(x,y) + \frac{n}{x}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y)\Big(\frac{j}{n}-x\Big)n^2\cdot$$

$$\cdot\int\limits_{\frac{j}{n}}^{\frac{j+1}{n}}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(t,z)\sqrt{(t-x)^2+(z-y)^2}\,dtdz := f'_x(x,y) + \frac{n}{x}A_n(x,y)$$

for $n \in N$. Applying Hölder inequalities, we get for $n \in N$

$$|A_n(x,y)| \le$$

$$\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y)\Big|\frac{j}{n}-x\Big|n\Bigg\{\int\limits_{\frac{j}{n}}^{\frac{j+1}{n}}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}}\psi^2(t,z)[(t-x)^2+(z-y)^2]dtdz\Bigg\}^{1/2} \le$$

$$\le \{S_{n,n}^{\{1\}}((t-x)^2;x,y)\}^{1/2}\{S_{n,n}^{\{2\}}(\psi^2(t,z)(t-x)^2;x,y) +$$

$$+ S_{n,n}^{\{2\}}(\psi^2(t,z)(z-y)^2;x,y\}^{1/2}$$

and, by (13) – (15),

$$S_{n,n}^{\{2\}}(\psi^2(t,z)(t-x)^2;x,y) \le \big\{S_{n,n}^{\{2\}}(\psi^4(t,z);x,y)\big\}^{1/2}\big\{S_n^{\{2\}}((t-x)^4;x)\big\}^{1/2},$$

$$S_{n,n}^{\{2\}}(\psi^2(t,z)(z-y)^2;x,y) \le \big\{S_{n,n}^{\{2\}}(\psi^4(t,z);x,y)\big\}^{1/2}\big\{S_n^{\{2\}}((z-y)^4;y)\big\}^{1/2},$$

which by Lemma 1, (12) and $\psi(x,y)=0$ yield

$$\lim_{n\to\infty} nS_{n,n}^{\{2\}}(\psi^2(t,z)(t-x)^2;x,y) = 0,$$

$$\lim_{n\to\infty} nS_{n,n}^{\{2\}}(\psi^2(t,z)(z-y)^2;x,y) = 0.$$

From the above facts and by (17) we deduce that

(31) $$\lim_{n\to\infty} nA_n(x,y) = 0, \qquad \text{for every fixed} \quad (x,y)\in R_+^2.$$

Using (31) to (30), we obtain (26) for $i=2$.

The proof of (26) for $f \in C^1_{2;q_1,q_2}$ is identical. □

Acknowledgment. The authors are greatly indebted to the referee for his helpful comments concerning this paper.

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